Some challenging optimisation problems for logistic diffusive equations and numerical issues

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SOME CHALLENGING OPTIMISATION PROBLEMS FOR LOGISTIC DIFFUSIVE EQUATIONS AND NUMERICAL ISSUES

IDRISS MAZARI, GRÉGOIRE NADIN, AND YANNICK PRIVAT

Abstract. This chapter is dedicated to the study of a shape optimization problem occurring in population dynamics. We provide some answers to the generic question: How to optimally arrange some resources in an enclosure? Of course, the wording "optimally" refers to different criteria, which may be the survival ability of a given species or the total population size. We consider here a very simple model in which the evolution of the population is governed by the logistic diffusive equation parametrized by the so-called intrinsic growth rate of a species denoted $m(\cdot)$, which is the main optimization variable and models the favorable and unfavorable parts of the habitat. We investigate here two optimal design problems, each corresponding to a possible modelling of the issue above: the first one is related to the species persistence for large times. It boils down to the optimization of the principal eigenvalue associated with an elliptic operator with respect to the resources distributions $m(\cdot)$. The second one deals with steady-states of the aforementioned reaction-diffusion equation and aims at maximizing the total size of the population with respect to resources distributions. In our analysis, we mainly focus on qualitative properties of maximizers, and illustrate it with the help of numerical illustrations. We also highlight related open problems and interesting numerical issues that remain to be investigated.

Keywords: extremal eigenvalue problem, shape optimization, symmetrization techniques.
AMS classication: 49J15, 49K20, 49R05, 49M05.

Contents

1. Introduction and bio-mathematical background

1.1. **General introduction.** The study of population dynamics is now well-established as a field of paramount importance, both in life sciences and in mathematics. The first models for population dynamics, which originated in the seminal [18, 24] and were then studied at length, relied on a spatial homogeneity assumption, and the influence of spatial heterogeneity and of the shape of the domain remained elusive for some time. It was successfully used for the first time in an ecological modelling

Date: October 31, 2020.

framework in [40]. The first work addressing in a crude way the influence of the environment on population dynamics was the series of articles by Cantrell and Cosner [6, 8, 7, 9], in which it is proved among other results that, assuming the boundary of the environment is lethal then no population can survive if this environment is too small. This opened the way for a finer understanding of these mathematical and biological queries, and, since then, this line of research has been extremely active. In this Chapter, we will present some recent works that are part of the endeavour to grasp a fuller understanding of spatially heterogeneous equations. Here, spatial heterogeneity will be taken into account via the **resources distribution**, and the main focus is the following informal question:

What is the optimal way to spread resources in a domain?

More precisely, we investigate two main issues:

(1) the optimal survival ability: how can we spread resources so as to optimise the survival ability of a population? We will present the relevant results in Section 2.
(2) the maximal total population size: how should we design the resources distribution in order to maximise the total population size? We will present the main results in Section 3.

It should be noted that we will thus be investigating **optimization problems**; since, for such problems, obtaining a complete description of the solutions is in general impossible, we focus here on **qualitative properties** of these solutions (e.g symmetry properties or bang-bang property-see Section 1.1.3), and the related numerical issues.
Structure of the Chapter. In Section 1.1.1, we present the biological model that will be the central focus of this Chapter; the two optimization problems informally stated out above are formalized in Section 1.1.2, and the properties under investigation are described in Section 1.1.3. Results on the optimal survival ability are gathered in Section 2, while the ones devoted to the population size are presented in Section 3. We conclude this chapter by discussing, in Section 4, possible generalizations of the works presented here and state several open problems, both theoretical and numerical, that remain to be investigated.

1.1.1. *The main biological model.* We use the classical Fisher-KPP model [18, 24] which, as has been acknowledged [15, 40, 41], captures several of the essential features of population dynamics. In this model, the population is assumed to live an habitat Ω assumed to satisfy the regularity assumption:

$$(\mathcal{H}_{\text{reg}}) \qquad \Omega \text{ is a bounded connected domain in } \mathbb{R}^d,\ d \geq 1 \text{ with a Lipschitz boundary.}$$

Smoothness assumptions on Ω will be completed and specified for each result. We make the following assumptions:

(1) The number of individuals is large enough that the population can be modelled as a density $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$,
(2) There is a non-linear crowding effect $-u^2$ which amounts for intra-specific competition,
(3) The population can access resources. These resources are modelled through a function $m \in L^\infty(\Omega)$, taken into account *via* a reaction term mu. The subset $\{m > 0\}$ corresponds to a favorable zone, while the zone $\{m \leq 0\}$ corresponds to a letal region.
(4) The population disperses at random in the domain with a characteristic dispersal rate $\sqrt{\mu}$. This leads to the diffusion term $\mu\Delta u$. We mention results related to other types of dispersal (i.e that take into account the spatial heterogeneity) in Section 4.
(5) Boundary conditions will be either of Dirichlet or Neumann type, i.e
$$u = 0 \text{ on } \partial\Omega \qquad \text{or} \qquad \partial_n u = 0 \text{ on } \partial\Omega.$$
We will moreover point to results for Robin boundary conditions in Section 4.
(6) We consider a non-trivial initial condition, i.e a nonnegative function u_0 in Ω, $u_0 \neq 0$.

Let us introduce the boundary conditions operator B defined by

$$Bu = u \text{ in the Dirichlet case,} \qquad Bu = \partial_n u \text{ in the Neumann case.}$$

The full evolution equation then reads

$$\text{(1)}\qquad \begin{cases} u_t(t,x) = \mu\Delta u(t,x) + u(t,x)[m(x) - u(t,x)] & (t,x) \in \mathbb{R}_+ \times \Omega, \\ Bu(t,x) = 0 & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x) & x \in \overline{\Omega}, \end{cases}$$

Of particular importance to understand the asymptotic behavior of the solution u of (1), as $t \to +\infty$, is the existence of **non-trivial steady states**, that is, of solutions $\theta_{m,\mu}$ of the equation

$$\text{(2)}\qquad \begin{cases} \mu\Delta\theta_{m,\mu}(x) + \theta_{m,\mu}(x)(m(x) - \theta_{m,\mu}(x)) = 0 & x \in \Omega, \\ B\theta_{m,\mu}(x) = 0 & x \in \partial\Omega, \\ \theta_{m,\mu}(\cdot) \geq 0\,, \theta_{m,\mu}(\cdot) \not\equiv 0. \end{cases}$$

Indeed, we expect that, as $t \to \infty$, the solution $u(t,\cdot)$ of (1) will converge to such steady-states. As it turns out, the question of existence of solutions $\theta_{m,\mu}$ of (2) and of convergence of $u(t,\cdot)$ to such steady states can both be solved by the investigation of a simple **spectral criterion**. Considering the linearization of (2) around $z \equiv 0$ leads to considering the linear differential operator

$$\mathcal{L}^B_{m,\mu} : u \in \mathcal{D}(\mathcal{L}^B_{m,\mu}) \mapsto -\mu\Delta - mu, \qquad B \in \{D, N\},$$

where the supercript B stands for the boundary conditions: we will use $B = N$ for Neumann boundary conditions, and $B = D$ for Dirichlet boundary conditions, so that

$$\begin{aligned} \mathcal{D}(\mathcal{L}^D_{m,\mu}) &= W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \\ \mathcal{D}(\mathcal{L}^N_{m,\mu}) &= \{u \in W^{2,2}(\Omega) \mid \partial_n u = 0 \text{ on } \partial\Omega\} \quad \text{(if } m \text{ does not vanish identically).} \end{aligned}$$

Consider the first eigenvalue $\lambda^B(m,\mu)$ of $\mathcal{L}^B_{m,\mu}$. We recall that it is defined through the minimization of a Rayleigh quotient:

$$\text{(3)}\qquad \lambda^D(m,\mu) := \inf_{u \in W^{1,2}_0(\Omega)\setminus\{0\}} \mathfrak{R}_{(m,\mu)}[u] \quad \text{and} \quad \lambda^N(m,\mu) := \inf_{u \in W^{1,2}(\Omega)\setminus\{0\}} \mathfrak{R}_{(m,\mu)}[u].$$

where

$$\mathfrak{R}_{(m,\mu)}[u] = \frac{\mu\int_\Omega |\nabla u|^2 - \int_\Omega mu^2}{\int_\Omega u^2}$$

This eigenvalue is well defined whenever $m \in L^\infty(\Omega)$. It is often called *principal eigenvalue* meaning that its associated eigenfunction, often called itself *principal eigenfunction*, does not change sign. The link with the existence of solutions of (2) and the convergence of solutions (1) was first observed in [39] and formalized in the following form in [5]: let $B \in \{D, N\}$.

(1) First case: $\lambda^B(m,\mu) < 0$. There exists a unique solution $\theta_{m,\mu}$ of (2), called *persistence steady-state*. Moreover, any solution $u = u(t,x)$ of (1) associated with a non-zero initial condition u_0 converges in $C^1(\overline{\Omega})$ to $\theta_{m,\mu}$ In other words, **the population survives**. Moreover, if $B = N$ (Neumann case), by taking $u(\cdot) = 1$ in the Rayleigh quotient formulation, we note that this condition is fulfilled provided $\int_\Omega m > 0$.
(2) Second case: $\lambda^B(m,\mu) \geq 0$. In that case, no solution of (2) exists. Any solution $u = u(t,x)$ of (1) associated with a non-zero initial condition u_0 converges in $C^1(\overline{\Omega})$ to 0, called *extinction steady-state*. In other words, **the population goes extinct**.

A possible interpretation of these results is that the eigenvalue $\lambda^B(m,\mu)$ with $B \in \{D, N\}$ quantifies the **survival ability** of the population.

Regarding the total population size, we note that if $\lambda^N(m,\mu) < 0$, uniqueness of the solution of (2) allows us to define the **total population size**

$$\text{(4)}\qquad F(m,\mu) = \int_\Omega \theta_{m,\mu}.$$

In what follows, we will not consider the Dirichlet case when dealing with the total population size (4). Indeed, our methods rely on an asymptotic analysis of the solutions as $\mu \to +\infty$. However, if $m(\cdot) \in L^\infty(\Omega)$ is given, for large values of μ (namely $\mu \geq \|m\|_{L^\infty}/\lambda^D_1(\Omega)$ where $\lambda^D_1(\Omega)$ denotes the first Dirichlet eigenvalue on Ω), one has $\lambda^D(m,\mu) \geq 0$, meaning that the stationary steady-state $\theta_{m,\mu}$ is trivial.

1.1.2. *The optimization problems.* The previous considerations led several authors [5, 23, 26, 29, 33, 37, 35] to consider optimization problems related to the survival ability and the total population size. Such problems are set under natural constraints, the first of which deals with the total amount of available resources: we introduce a fixed parameter $m_0 \in (0,\kappa)$ and require that the resources distributions m considered satisfy

$$\int_\Omega m = m_0|\Omega|.$$

This accounts for the fact that, in a given domain, only a limited amount of resources is available. The second constraint is a pointwise one, and accounts for natural limitations of the environment, i.e the fact that, in a single spot, only a maximum amount of resources may be available. This is modelled by introducing a real parameter $\kappa > 0$ and requiring that admissible resources distributions m satisfy

$$\|m\|_{L^\infty(\Omega)} \leq \kappa.$$

In this chapter, we will further simplify this constraint and require the following stronger assumption:

$$0 \leq m \leq \kappa \quad \text{a.e. in } \Omega.$$

For optimization problems without such a constraint, see for instance [2]. In Section 4 we provide some comments on sign-changing resources distributions. In other words, the admissible class is

$$\mathcal{M}(\Omega) := \left\{ m \in L^\infty(\Omega)\,, \int_\Omega m = m_0|\Omega|,\ 0 \leq m \leq \kappa \right\},$$

and we implicitly assume $\kappa > m_0$. As was noted, the condition $m \in \mathcal{M}(\Omega)$ ensures that $\lambda(m,\mu) < 0$, and so $\theta_{m,\mu}$ is indeed well-defined.

The first optimization problem deals with the optimal survival ability. It writes

$$\textbf{(I)} \qquad \boxed{\inf_{m\in\mathcal{M}(\Omega)} \lambda^B(m,\mu).} \qquad (B \in \{D,N\})$$

The relevant results for this problem are gathered in Section 2, but let us note that this optimization problem ultimately deals with an evolution problem, as it governs the long-time behaviour of the evolution equation (1).

Note that most of the results available in the Neumann case extend to periodic boundary conditions, using the arguments developed in [38].

The second optimization problem deals with the total population size, and reads

$$\textbf{(II)} \qquad \boxed{\sup_{m\in\mathcal{M}(\Omega)} \int_\Omega \theta_{m,\mu}.}$$

where $\theta_{m,\mu}$ solves (2) with Neumann boundary conditions.

We present the available results in Section 3. This optimization problems involve the unique solution of a stationary elliptic equation.

1.1.3. *Type of properties investigated.* Let us now state which properties will be investigated. Whether it be spectral optimisation or optimal control problems for non-linear equations, obtaining a complete description of maximizers is in general impossible. In this Chapter, we focus on *qualitative properties* of maximizers: following the seminal works [5, 39], we are looking for general paradigms of the form: *Maximizers should satisfy the bang-bang property* or *Concentrating resources is favourable for survival of species* and, in general, for (non)-symmetry of optimizers.

Regarding the survival ability of species, we mainly follow the lines of [26] in order to address such intricate questions. Regarding the total population size, we will mostly follow, for our presentation, the articles [33, 35, 37].

Let us now present these two properties.

The bang-bang property: the first relevant property is the so-called **bang-bang property**. Namely, since we are working with L^∞ constraints, are the solutions of (**I**) and (**II**) of the form $m^* = \kappa \mathbb{1}_E$ for some subset $E \subset \Omega$? When such a property is satisfied, which is always the case for (**I**), it proves to be very convenient for numerical applications. We note that, for the spectral optimization problem (**I**), this property is an easy consequence of the concavity of the functional under consideration and of the convexity of the set of admissible resources distributions, see Theorem 1.

Fragmentation and concentration of resources: other particularly relevant features of (**I**) and (**II**) are *concentration* and, conversely, *fragmentation* of resources. To explain these features, let us assume that some maximizers $m_{\mathbf{I}}$ and $m_{\mathbf{II}}$ of, respectively, (**I**) and (**II**) satisfy the bang-bang properties. In other words, there exist two subsets $E_{\mathbf{I}}$ and $E_{\mathbf{II}}$ such that

$$m_{\mathbf{I}} = \kappa \mathbb{1}_{E_{\mathbf{I}}}, \quad m_{\mathbf{II}} = \kappa \mathbb{1}_{E_{\mathbf{II}}}.$$

What do $E_{\mathbf{I}}$ and $E_{\mathbf{II}}$ look like? How many connected components do they have? If the set Ω enjoys some symmetry properties, do $E_{\mathbf{I}}$ and $E_{\mathbf{II}}$ enjoy the same symmetries?

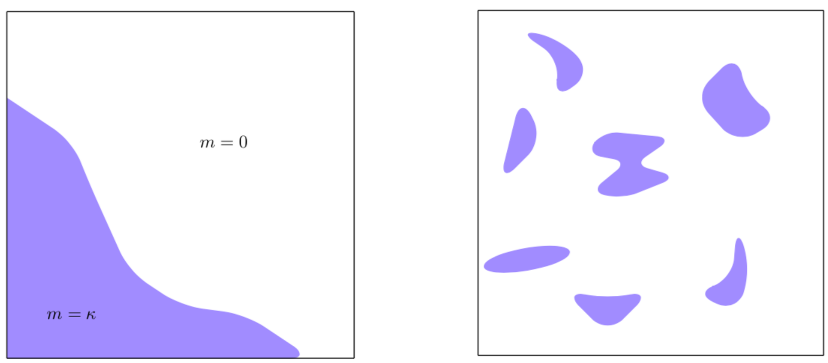


FIGURE 1. $\Omega = (0,1)^2$. The resources distribution on the left is "more concentrated" than the one on the right.

In the case of spectral optimization (**I**) this property is known to hold in certain sets, for instance in orthotopes $\Omega = (0,1)^n$, and corresponds to the paradigm that *concentrating resources favors survival* [5, 39]. This is essentially due to the *energetic nature of spectral functionals*, which allows for the use of rearrangement inequalities. For the total population size however, no such results were available until [33, 35], and we will show that the answer to this question is, for this problem, highly dependent on the rate of the characteristic dispersal rate: concentration does hold for large diffusivities μ, while fragmentation does for small diffusivities.

Let us mention that effects of movement and spatial heterogeneity on population dynamics via reaction-diffusion-advection models, focusing on the persistence, competition, and evolution of organisms in spatially heterogeneous environments are much discussed in the recent survey article [25].

2. Optimal eigenvalue problem

This section is devoted to the analysis of Problem (**I**).

2.1. **Qualitative analysis.** The first natural issue, when dealing with Problem (**I**) is related to the existence and *bang-bang* property of minimizers. It has been investigated in particular in [21, 22, 32, 38] for several boundary conditions and in several dimensions. More recently, an elegant argument has been proposed in [16], yielding at the same time the existence of a *bang-bang minimizer* m^* and that the minimizing set $\{m^* = \kappa\}$ is a level surface of the principal eigenfunction. We reproduce this argument hereafter in the case of Neumann boundary conditions.

Theorem 1. *Let $B \in \{D, N\}$. Problem (I) has a solution m^*, and moreover there exists a measurable subset $E^* \subset \Omega$ such that, up to a set of zero Lebesgue measure, there holds*

$$m^* = m_{E^*} \quad \text{where} \quad m_{E^*} = \kappa \mathbb{1}_{E^*} \text{ a.e. in } \Omega,$$

Moreover, let u denotes a nonnegative eigenfunction (nique up to a multiplicative constant) associated to the eigenvalue problem (3). There exists $\mu > 0$ such that $E^* = \{u \geq \mu\}$.

Proof. Let us assume without loss of generality that $B = N$, the proof in the Dirichlet case being an immediate adaptation of what follows. Observe first that the existence is standard, see for instance [20, Thm 8.1.2] where it is shown that $m \mapsto \lambda^N(m)$ is continuous for the L^∞ weak-$\star$ topology and the set of admissible weights $\mathcal{M}_{m_0,\kappa}$ is compact for this topology. Notice furthermore that, as the infimum of linear functionals, $m \mapsto \lambda^N(m)$ is concave.

Let m^* be a minimizer for Problem (I) and denote by u its associated eigenfunction. Direct computations show that

$$\lambda^N(m^*) \;=\; \frac{\mu \int_\Omega |\nabla u|^2 - \int_\Omega m^* u^2}{\int_\Omega u^2} \geq \frac{\mu \int_\Omega |\nabla u|^2 - \sup_{\tilde{m} \in \mathcal{M}(\Omega)} \int_\Omega \tilde{m} u^2}{\int_\Omega u^2},$$

According to the so-called *bathtub principe* (see for instance [27, Theorem 1.14]), there exists a measurable subset $E^* \subset \Omega$ such that

$$\sup_{\tilde{m} \in \mathcal{M}(\Omega)} \int_\Omega \tilde{m} u^2 = \kappa \int_\Omega \mathbb{1}_{E^*} u^2$$

and

$$\{u > t\} \subset E^* \subset \{u \geq t\} \quad \text{and} \quad |E^*| = m_0 |\Omega| / \kappa.$$

for a given $t > 0$. Observe moreover that E^* is defined in a unique way since the level sets of u have zero measure (see e.g. [28, Corollary 1.1]). We thus infer that

$$\lambda^N(m^*) \geq \frac{\mu \int_\Omega |\nabla u|^2 - \kappa \int_\Omega \mathbb{1}_{E^*} u^2}{\int_\Omega u^2} \geq \lambda^N(\kappa \mathbb{1}_{E^*}).$$

By minimality of $\lambda^N(m^*)$, it follows that all inequalities above are in fact equalities. Furthermore, since all the level sets of the eigenfunction u have zero Lebesgue measure, one easily shows that if m^* is not *bang-bang*, the first inequality above is strict, leading to a contradiction. The expected conclusion follows. $\square$

Let us now comment on the regularity of the optimal set E^* introduced in the statement of Theorem 1.

Remark 1 (On the regularity of the free boundary)**.** *Proving the regularity of the free boundary $\Gamma := \partial E^* \backslash \partial \Omega$ is a very difficult question in general. It follows from classical elliptic regularity that the principal eigenfunction u is $\mathcal{C}^{1,a}(\Omega)$ for every $a \in [0,1)$. Hence, as $E^* = \{u > \alpha\}$ up to a set of Lebesgue measure zero, the boundary Γ is $C^{1,a}$-smooth at any point where $\nabla u \neq 0$ and therefore, using a bootstrap argument, one infers the local analytic regularity of Γ in this case, see [12]. The regularity problem is thus reduced to the one of the degeneracy of the eigenfunction u on its level line Γ.*

When Dirichlet conditions are imposed on the boundary $\partial\Omega$, then it has been proved in [14], when $d = 2$, that $u \in \mathcal{C}^{1,1}(\Omega)$, that ∂E does not hit the boundary and consists of finitely many disjoint, simple and closed real-analytic curves. In higher dimensions, it is only known that Γ is smooth up to a closed set of Hausdorff dimension $d - 1$ [13]. However, the situation is much more complicated since one could expect, as for some other free boundary problems, the emergence of stable singularities.

Before focusing on symmetry properties of optimizers, let us highlight that all solutions are known in the simple one-dimensional case, namely $d = 1$.

Proposition 1. *Let us assume that $\Omega = (0,1)$, $\kappa > 0$ and $m_0 \in (0, \kappa)$.*

- *Dirichlet case. Problem* (I) *has a unique solution* m^* *corresponding to a concentrated and centered resources distributions, namely:*
$$m^* = \kappa \mathbb{1}_{((1-m_0)/2,(1+m_0)/2)}.$$
- *Neumann case. Problem* (I) *has exactly two solutions* m_1^* *and* m_2^* *corresponding to concentrated resources distributions meeting the boundary of* Ω*, namely*
$$m_1^* = \kappa \mathbb{1}_{(0,m_0)} \qquad \text{and} \qquad m_2^* = m_1^*(1-\cdot).$$

This result is illustrated on Fig. 2 below.

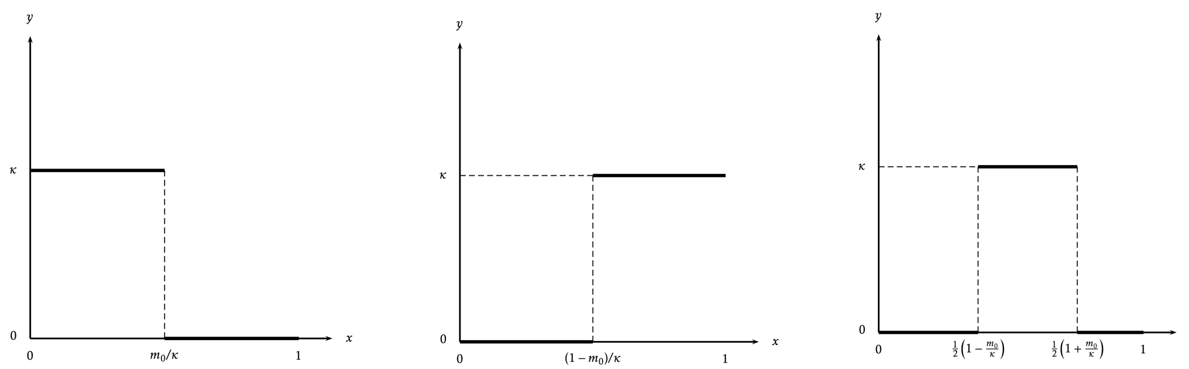


FIGURE 2. $\Omega = (0,1)$. Left and middle: optimal resources distribution in the Neumann case. Right: optimal resource distribution in the Dirichlet case.

The proof rests upon an adequate symmetrization argument.

The following theorem is dedicated to the analysis of optimal domains for Problem (I), when dealing with Dirichlet boundary conditions. It follows from an easy adaptation of the results from [11] and [26].

Theorem 2 (Dirichlet case)**.** *Let us assume that* $B = D$, $d \geq 2$ *and* $\kappa > 0$. *Let* E^* *be a subset of* Ω *such that* $m^* = \kappa \mathbb{1}_{E^*}$ *solves Problem* (I)*. Then,*

- *the complement set of* E^* *in* Ω *contains a tubular neighborhood of the boundary* $\partial\Omega$*;*
- *there exists* $\mu_0(\Omega, m_0, \kappa) > 0$ *such that if* $\mu \geq \mu_0(\Omega, m_0, \kappa)$*, then every connected component of the complement set of* E^* *hits the boundary* $\partial\Omega$*.*
- *assume that the domain* Ω *is symmetric and convex with respect to the hyperplane* $\mathscr{H} := \{x_1 = 0\}$*. Then,* E^* *is symmetric and convex with respect to with respect to* $\mathscr{H}$*. Furthermore, the associated nonnegative eigenfunction* u *is decreasing in* x_1 *for* $x_1 \geq 0$*.*
- *if one assumes that* Ω *is convex and has a* $\mathscr{C}^2$ *boundary, there exists* $\mu_0(\Omega, m_0, \kappa) > 0$ *such that if* $\mu \geq \mu_0(\Omega, m_0, \kappa)$*, then* E^* *is convex and* $\partial E^* \cap \Omega$ *is real analytic.*
- *if* E^* *or* $\Omega \backslash E^*$ *is rotationally symmetric (i.e. a union of concentric rings, whose center is denoted O) and has a finite number of connected components, then* E^* *and* Ω *are concentric balls.*
- *if* Ω *is a ball centered at* O*, then so is* E^**.*

The next result is dedicated to the analysis of optimal domains for Problem (I), when dealing with Neumann boundary conditions. A common conjecture in dimension 2 in this framework is that the minimizing set has constant curvature, that is, it would be a quarter of ball, a stripe, or the complementary of a quarter of ball depending on the parameters (see e.g. [23, 38] and Figure 4 hereafter). The results above contradict this conjecture and are issued from [26, Section 5].

Theorem 3 (Neumann case)**.** *Let us assume that* $B = N$, $d \geq 2$ *and* $\kappa > 0$. *Let* E^* *be a subset of* Ω *such that* $m^* = \kappa \mathbb{1}_{E^*}$ *solves Problem* (I)*.*

- *If* $\Omega = \Pi_{k=1}^N (0, L_k)$*, then*

(1) (Steiner symmetry) m_{E^*} is monotonic with respect to $x_k \in (0, L_k)$ for all k.
(2) if $\partial E^* \cap \Omega$ is analytic, then $\partial E^* \cap \Omega$ does not contain any piece of sphere.

- *If $\Omega = B(0,1)$, then*
  (1) (Circular Symmetry) there exists $\theta_0 \in [0, 2\pi)$ such that E^* is symmetric with respect to the half straight line $\{\theta = \theta_0\}$ in the radial coordinates (r, θ). Moreover, for all $r \in (0,1)$, $\{\theta \in [0, 2\pi), (r, \theta) \in E^*\}$ is an interval.
  (2) If $\beta = 0$, then E^* is not a ball.

2.2. **Numerical investigations.** Let us first introduce the numerical algorithm we used for computing optimal sets. This approach has been first introduced in [11] and used in [21, 26]. It strongly rests upon the characterization of the optimal set E^* provided in Theorem 1, as a level set of the principal eigenfunction u. Let us explain the method's principle before summarizing the complete algorithm. The aim is to implement a kind of fixed point procedure resting upon the formula

$$E^* = \{u_{E^*} \geq \mu_{E^*}\}$$

where E^* denotes an opimizer for Problem (**I**), u_{E^*} its associated eigenfunction (uniquely chosen to be nonnegative and normalized in $L^2(\Omega)$) and $\mu_{E^*} > 0$ the corresponding Lagrange multiplier associated to the volume constraint of E^*, see Theorem 1.

Consider an arbitrary set E_k with measure $m_0|\Omega|/\kappa$, and its associated eigenfunction u_{E_k} (uniquely defined as above). Then, since the Lagrange multiplier can be seen as a parameter adjusted to preserve the volume constraint, we update E_k by setting

$$E_{k+1} = \{u_k \geq \mu_k\},$$

where $\mu_k > 0$ denotes the unique number such that $|\{u_k \geq \mu_k\}| = m_0|\Omega|/\kappa$.

Algorithm 1: Fixed point procedure to compute local optimizers E^*_{num}

Initialization:

Let E_0 be an arbitrary subset of Ω, maxiter $\in \mathbb{N}^*$ and $\varepsilon_{\text{tol}} > 0$ be given;
Set $k = 0$ and $\delta = \varepsilon_{\text{tol}} + 1$;
Compute (λ_0^B, u_0), the principal eigenpair associated to E_0, with u_0 chosen to be nonnegative and $L^2(\Omega)$ normalized;

While $k \leq$ maxiter *and* $\delta > \varepsilon_{\text{tol}}$ **do**

1: compute $\mu_k > 0$ so that $|\{u_k \geq \mu_k\}| = m_0|\Omega|/\kappa$ by implementing a standard bissection method;
2: compute the level set $E_{k+1} = \{u_k \geq \mu_k\}$;
3: compute $(\lambda_{k+1}^B, u_{k+1})$, the principal eigenpair associated to E_{k+1}, with u_{k+1} chosen to be nonnegative and $L^2(\Omega)$ normalized;
4: update $\delta \leftarrow |\lambda_{k+1} - \lambda_k|$

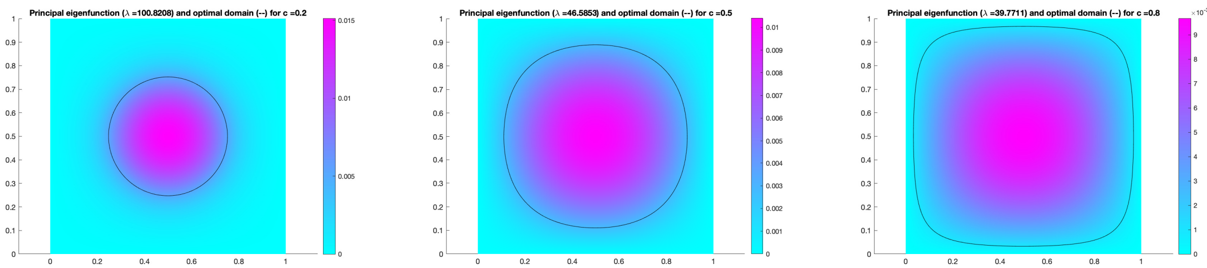
End:

Set $E^*_{\text{num}} = E_k$.

It is notable that this leads to a descent algorithm, as observed in [21], although there does not exist any result regarding the complete convergence analysis up to our knowledge.

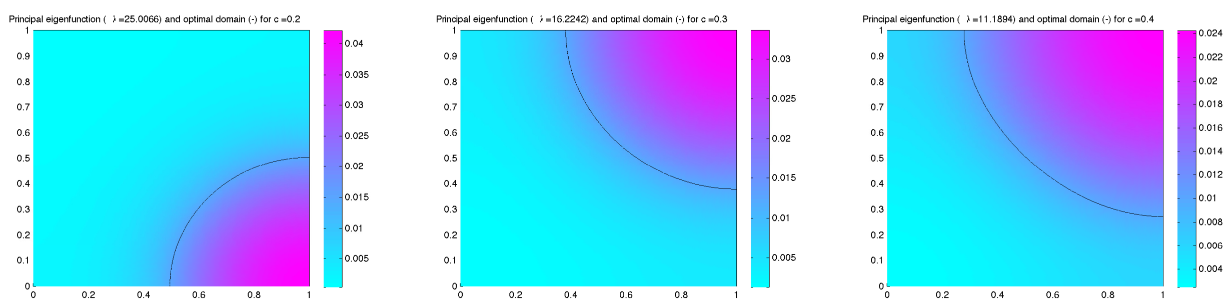
This method has been used to compute optimal sets for Dirichlet boundary conditions [11], Neumann boundary conditions in squares and ellipses [23], Robin boundary conditions in squares [21]. In general these solutions look like stripes, balls, or complementary of balls, depending on the parameters.

Numerical results for the square and the ball in the Neumann case are gathered on Figures 4 and 6. Convergence curves illustrating the efficiency of the method are drawn on Figures 5 and 7. On Figure 8, we provide additional examples in the case of an ellipsis with semi-axis 1 and $1/\sqrt{\pi}$.

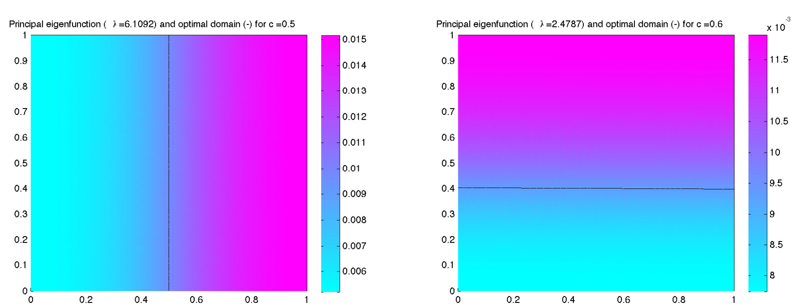


(a) $|E| = 0.2$ (b) $|E| = 0.5$ (c) $|E| = 0.8$

FIGURE 3. $\Omega = (0,1)^2$. Optimal domains in the Dirichlet case with $\kappa = 0.5$ and several volume constraints on E. Colors correspond to the intensity levels of the eigenfunction and the domain E encloses the zone corresponding to the warmest colors. Note that we know by Theorem 2 that the boundaries of the patterns observed in (a), (b) and (c) do not contain any part with constant curvature.

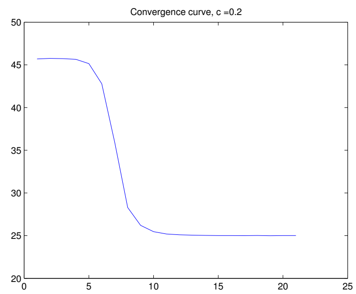


(a) $|E| = 0.2$ (b) $|E| = 0.3$ (c) $|E| = 0.4$

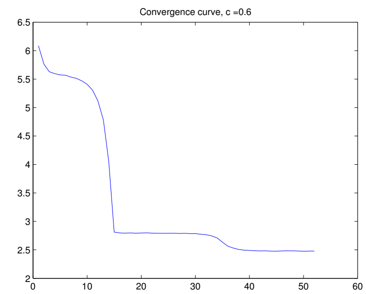


(d) $|E| = 0.5$ (e) $|E| = 0.6$

FIGURE 4. $\Omega = (0,1)^2$. Optimal domains in the Neumann case with $\kappa = 0.5$ and several volume constraints on E. Colors correspond to the intensity levels of the eigenfunction and the domain E encloses the zone corresponding to the warmest colors. Note that we know by Theorem 3 that the boundaries of the patterns observed in (a), (b) and (c) do not contain any part with constant curvature.

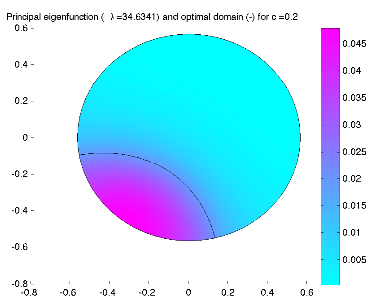


(a) $|E| = 0.2$ - convergence curve

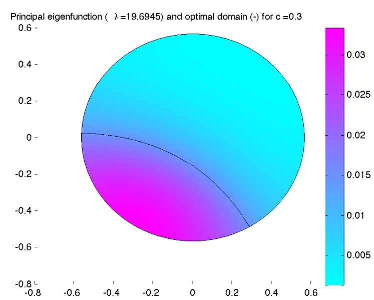


(b) $|E| = 0.6$ - convergence curve

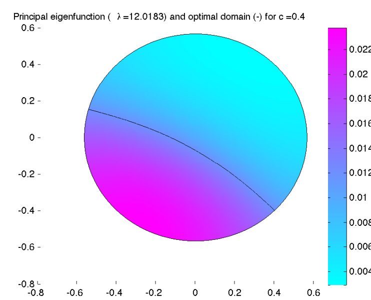
Figure 5. $\Omega = (0,1)^2$. Two examples of convergence curves in the Neumann case ($\beta = 0$) with $\kappa = 0.5$, $c = 0.2$ (left) and $c = 0.6$ (right)



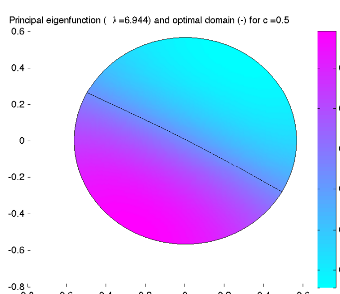
(a) $|E| = 0.2$



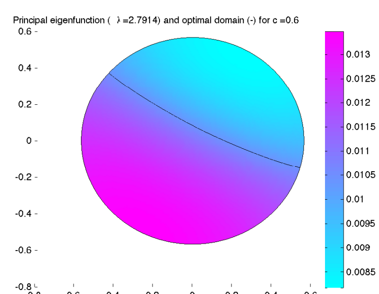
(b) $|E| = 0.3$



(c) $|E| = 0.4$



(d) $|E| = 0.5$



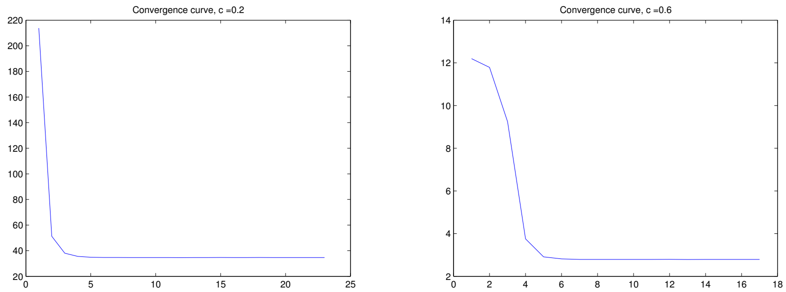
(e) $|E| = 0.6$

Figure 6. $\Omega = B(0, 1/\sqrt{\pi})$. Optimal domains in the Neumann case with $\kappa = 0.5$ and several volume constraints on E. Colors correspond to the intensity levels of the eigenfunction and the domain E encloses the zone corresponding to the warmest colors.

3. Maximizing the total population size

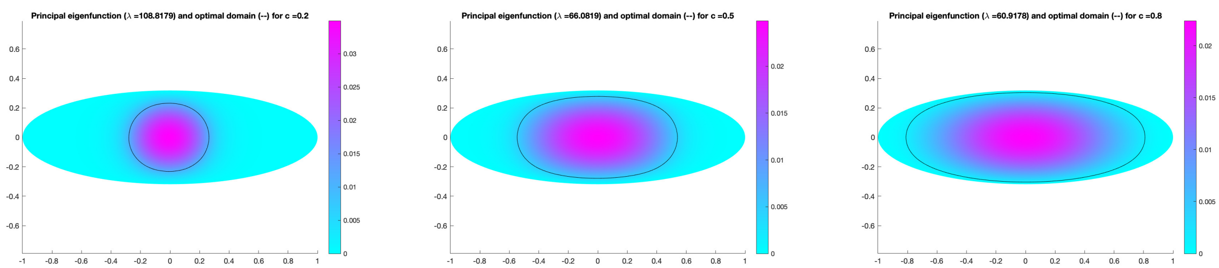
This section is devoted to the analysis of Problem (II).

3.1. **Qualitative analysis.** We state here two important properties of solutions of (II), whose proof can be found in [33].
In what follows, we will assume that $\Omega \subset \mathbb{R}^d$, with $d \in \mathbb{N}^*$ is a connected open set with a $\mathscr{C}^2$ boundary, or that $\Omega = (0,1)^d$.

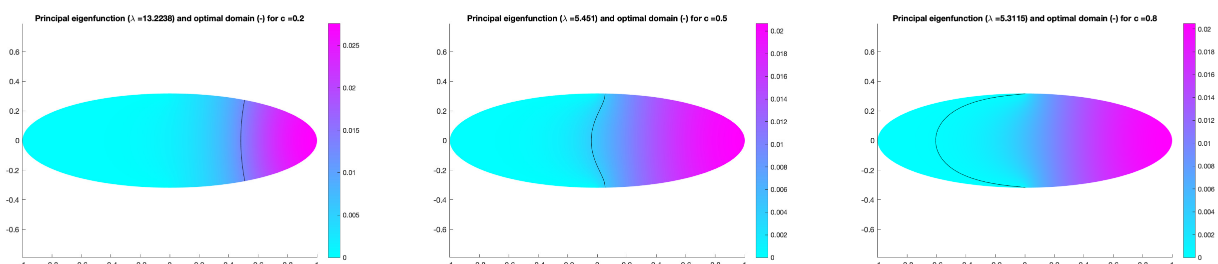


(a) $c = 0.2$ - convergence curve (b) $c = 0.6$ - convergence curve

FIGURE 7. $\Omega = B(0, 1/\sqrt{\pi})$. Two examples of convergence curves in the Neumann case with $\kappa = 0.5$ and $|E| = 0.2$ (left) or $|E| = 0.6$ (right)



(a) $|E| = 0.2$ (b) $|E| = 0.5$ (c) $|E| = 0.8$



(d) $|E| = 0.2$ (e) $|E| = 0.5$ (f) $|E| = 0.8$

FIGURE 8. $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2/\pi = 1\}$. Top: optimal domains in the Dirichlet case with $\kappa = 0.5$ and several volume constraints on E. Bottom: optimal domains in the case of mixed Dirichlet-Neumann boundary conditions (Dirichlet on the left part of the ellipsis and Neumann in the right one) with $\kappa = 0.5$ and several volume constraints on E. Colors correspond to the intensity levels of the eigenfunction and the domain E encloses the zone corresponding to the warmest colorscurvature.

Recall that our aim is to solve Problem (II)

$$\sup_{m \in \mathcal{M}(\Omega)} F_\mu(m)$$

where the functional F_μ is defined by

$$F_\mu(m) = \int_\Omega \theta_{m,\mu},$$

and $\theta_{m,\mu}$ solves (2).

This problem has been stated by Lou in his survey article [30]. The main difficulty for establishing this result rests upon the facts that $\theta_{m,\mu}$ solves a nonlinear PDE and the criterion F_μ does not derive from an energy. It has been partially addressed in [17], in which the authors apply the so-called Pontryagin principle, show the Gâteaux-differentiability of the functional and carry out a few numerical simulations backing up the conjecture that its maximizers are of bang-bang type, in other words equal to 0 or κ a.e. in Ω.

However, proving this bang-bang property is challenging. The analysis of optimality conditions appears rather intricate. Indeed, the sensitivity of the total population size functional with respect to the variations of $m(\cdot)$ is directly related to the solution of the adjoint state, i.e. the solution of:

$$\begin{cases} \mu\Delta p_{m,\mu} + p_{m,\mu}(m - 2\theta_{m,\mu}) = 1 & \text{in } \Omega, \\ \frac{\partial p_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Note that $p_{m,\mu}$ belongs to $W^{1,2}(\Omega)$ and is unique, according to the Fredholm alternative. Then, one can prove that, for all $\mu > 0$, the application F_μ is Gâteaux-differentiable with respect to m in direction h and its Gâteaux derivative writes

$$dF_\mu(m)[h] = -\int_\Omega h\theta_{m,\mu}p_{m,\mu}.$$

Now consider a maximizer m^*. To derive optimality conditions, we introduce, for a given $m \in \mathcal{M}(\Omega)$, the cone $\mathcal{T}_{m,\mathcal{M}(\Omega)}$ of admissible perturbations at m, namely the set of functions $h \in L^\infty(\Omega)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(\Omega)$ converging to h as $n \to +\infty$, and $m + \varepsilon_n h_n \in \mathcal{M}_{m_0,\kappa}(\Omega)$ for every $n \in \mathbb{N}$.

If m^* is a maximiser, then for every perturbation h in the cone of admissible perturbations $\mathcal{T}_{m^*,\mathcal{M}(\Omega)}$, there holds $dF_\mu(m^*)[h] \geq 0$. The analysis of such optimality condition leads to the following result.

Proposition 2. *Let us define $\varphi_{m,\mu} = \theta_{m,\mu}p_{m,\mu}$, where $\theta_{m,\mu}$ and $p_{m,\mu}$ solve respectively equations (2) and (5). There exists $c \in \mathbb{R}$ such that*

$$\{\varphi_{m,\mu} < c\} = \{m = \kappa\}, \quad \{\varphi_{m,\mu} = c\} = \{0 < m < \kappa\}, \quad \{\varphi_{m,\mu} > c\} = \{m = 0\}.$$

This property shows that exploiting properties of optimal configurations needs hence a deep understanding of the behavior of $\theta_{m,\mu}$ as well as that of the adjoint state. Using an analogous property, Nagahara and Yanagida, [37] proved that if the optimal resources distribution is Riemann Integrable then it is of bang-bang type. Their proof is valid for all $\mu > 0$, while the result we will present below in Theorem 4 is only valid for large μ's. But their regularity hypothesis might be restrictive. Indeed, we will display in Section 3.2 some numerical simulations showing that the maximizer might oscillate a lot when μ becomes small.

Property no. 1: pointwise constraints, bang-bang property. In order to overcome these difficulties, we introduced in [33] a new method based on series expansions in powers of the diffusivity μ asymptotic in order to work out optimality conditions.

Theorem 4 ([33], Theorem 1)**.** *Let $\mu > 0$, $\kappa > 0$, $m_0 \in (0, \kappa)$. There exists a positive number $\mu^* = \mu^*(\Omega, \kappa, m_0)$ such that, for every $\mu \geq \mu^*$, the functional F_μ is strictly convex. As a consequence, for $\mu \geq \mu^*$, every maximizer of Problem (**II**) is bang-bang, that is, $m^* = \kappa\mathbb{1}_E$, where $\mathbb{1}_E$ is the characteristic function of a (measurable) resources set E.*

The proof rests upon a tricky expansion of the solution $\theta_{m,\mu}$ of (2), as a series involving the solutions of a sequence of cascade systems.

Property no. 2: concentration-fragmentation of maximizers. It is well-known (see e.g. [5, 26]) that concentrating resources, meaning that the resources distribution m is decreasing in each direction, favors the survival of species. On the contrary, we will say that a resources set is fragmented whenever it is disconnected (see Fig. 1).

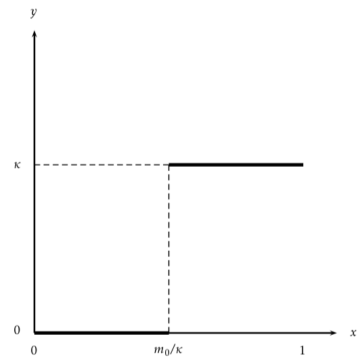


FIGURE 9. A solution of (II) in 1D, for large diffusivities μ.

Theorem 5 ([33], Theorem 2)**.** *Consider $\Omega = (0,1)^d$. Any family of maximizers $\{m^*_\mu\}_{\mu>0}$ for Problem (II) converges in $L^1(\Omega)$ to the characteristic function of a set E which is concentrated (meaning that its characteristic function $\mathbb{1}_E$ is monotone with respect to each space variable).*

In the one-dimensional case, we also prove that if the diffusivity is large enough, there are only two maximizers, that are simple crenels meeting either the left or the right boundary (see Fig. 9 above).

In order to prove these results, we show that the maximizers converge, when μ tends to $+\infty$, to the maximizers of a limit problem. Indeed, let us introduce the function space

$$X := W^{1,2}(\Omega) \cap \left\{ u \in W^{1,2}(\Omega)\,,\, \int_\Omega u = 0 \right\}$$

and the energy functional

$$\mathcal{E}_m : X \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - m_0 \int_\Omega mu.$$

Theorem 6. *([33], Γ-convergence property). For any $\mu > 0$, let m^*_μ be a solution of Problem (II). Any L^1 closure point of $\{m^*_\mu\}_{\mu>0}$ as $\mu \to \infty$ is a solution of the optimal design problem*

$$\min_{m \in \mathcal{M}_{m_0,\kappa}(\Omega)} \min_{u \in X} \mathcal{E}_m(u).$$

The above results are derived from an investigation of the minimization of $\mathcal{E}_m$.

We next obtain a surprising result: **fragmentation may be better than concentration for small diffusivities**.

Theorem 7 ([33], Theorem 4)**.** *Let $\Omega = (0,1)$. The function $\tilde{m} = \kappa\chi_{(1-\ell,1)}$ (and $\tilde{m}(1-\cdot) = \kappa\chi_{(0,\ell)}$) does not solve Problem (II) for small values of μ. More precisely, if we extend $\tilde{m}$ outside of $(0,1)$ by periodicity, there exists $\mu > 0$ such that*

$$\int_\Omega \theta_{\tilde{m},\mu} < \int_\Omega \theta_{\tilde{m}(2\ \cdot),\mu}.$$

This property has been investigated more precisely in [35]. Define for $M > 0$ the class

$$\mathcal{M}_M(\Omega) := \left\{ m \in \mathcal{M}(\Omega)\,,\, \|m\|_{BV(\Omega)} \leq M \right\}.$$

Theorem 8 ([35], Theorem 1)**.** *Consider a family of maximizers $\{m^*_\mu\}_{\mu>0}$ of Problem (II). There holds*

$$\|m^*_\mu\|_{BV(\Omega)} \xrightarrow[\mu \to 0^+]{} +\infty. \tag{6}$$

More precisely:

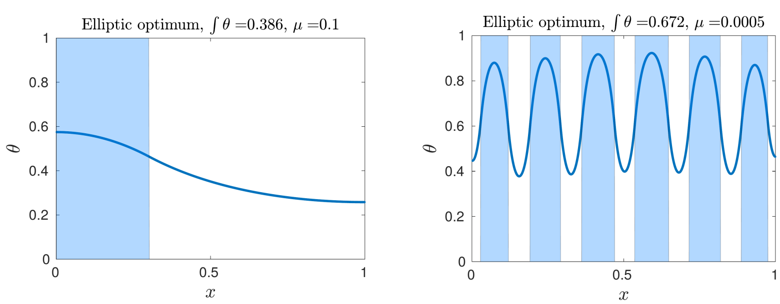
$$\forall M > 0, \quad \exists \mu_M > 0 \text{ s.t. } \forall\, 0 < \mu \leq \mu_M \sup_{m \in \mathcal{M}_M(\Omega)} \int_\Omega \theta_{m,\mu} < \sup_{m \in \mathcal{M}(\Omega)} \int_\Omega \theta_{m,\mu}. \tag{7}$$

Let us finally mention that a discretized version of this problem has been studied in [31], and has led to a very precise identification of discrete minimizers, highlighting that the number of related components increases when μ tends towards 0.

3.2. **Numerical investigations.** We provide hereafter several numerical simulations of optimizers of the total population size, solving Problem (**II**). These simulations have been obtained in the following way:

- in the optimization procedure described below, one encodes the control $m(\cdot)$ through its Fourier coefficients. To avoid the emergence of local maximizers, one picks several random initial guesses (by randomizing the first Fourier coefficients and applying an affine transformation to them to guarantee that $m(\cdot)$ satisfies the constraints);
- one works with a uniform space discretization.
- For each initial guess of $m(\cdot)$, one computes the solution θ of the logistic equation (2), and the solution to the adjoint-state (5) by using a finite differences method.
- We then implement a constrained adapted gradient type descent method, where the perturbation h of the control m is obtained by minimizing $h \mapsto dF_\mu(m)[h]$ over the set of all admissible perturbations $h(\cdot)$, namedy the set of all elements $h \in L^\infty(\Omega)$ such that $\int_\Omega h = 0$ and $m + h$ belongs almost everywhere to $[0, \kappa]$.
- This allows to select numerical optimizers and to perform a last gradient descent to ensure the robustness of the found solution.

We refer to [35] for further explanations on the employed method.



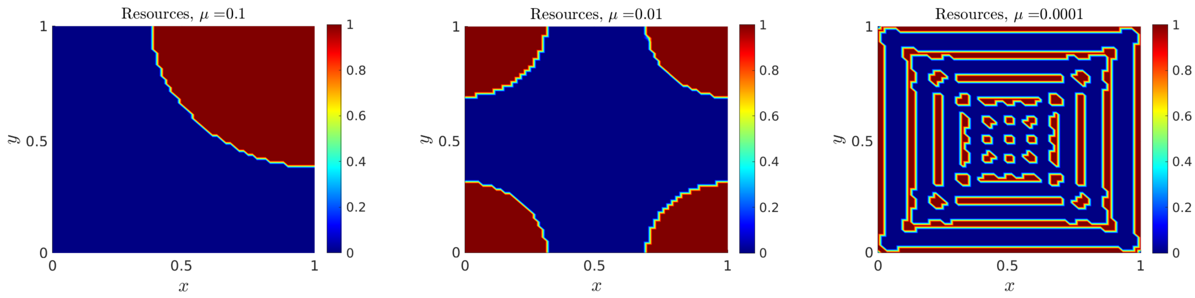
(a) (b)

FIGURE 10. A solution of (**II**) in 1D in pale, for small diffusivities μ, and the associated eigenfunction in dark blue.

4. Generalization and perspectives

In this chapter, we have summarized the known results related to the problem of maximizing population survival when controlled by a resource term. The problems considered in this chapter can be extended/generalized in various directions in a natural way. We discuss in the next section the generalization of the previous models to a drift operator and conclude this chapter by mentioning various problems that are still open or natural generalizations of the issues discussed here.

We mention nevertheless that we choose to not tackle the generalization to Problems (**I**) and (**II**) to other boundary conditions in the present article. Nevertheless, the analysis of such conditions when dealing with Problems (**I**) can be found in [26].



(a) (b) (c)

FIGURE 11. $\Omega = (0,1) \times (0,1)$. Optimal domains (in red) in the Neumann case with $\kappa = 1$, $m_0 = 0.3$ and various dispersal rates μ (according to [35]).

4.1. **Spectral optimisation for the biased movement of species.** For $\varepsilon \geq 0$, let us consider a population density with a resource term $m(\cdot)$ temporarily assumed to be differentiable and whose flux is $-\nabla u + \varepsilon u \nabla m$. The term $u\nabla m$ stands for a bias in the population movement, modeling a tendency of the population to disperse along the gradient of resources and hence move to favorable regions. The parameter ε quantifies the influence of the resources distribution on the movement of the species. The complete associated reaction diffusion equation, called "logistic diffusive equation", reads

$$\frac{\partial u}{\partial t} = \nabla \cdot \Big(\nabla u - \varepsilon u \nabla m\Big) + mu - u^2 \quad \text{in } \Omega,$$

completed with suitable boundary conditions. Plugging the change of variable $v = e^{-\varepsilon m} u$ in this equation leads to

$$\frac{\partial v}{\partial t} = \Delta v + \varepsilon \nabla m \cdot \nabla v + mv - e^{\varepsilon m} v^2 \quad \text{in } \Omega.$$

It is known (see e.g. [4, 3, 36]) that the asymptotic behavior of this equation is driven by the principal eigenvalue of the operator $\mathcal{L} : \psi \mapsto -\Delta\psi - \varepsilon \nabla m \cdot \nabla \psi - m\psi$. At this point, we are deliberately imprecise about the definition of the operator $\mathcal{L}$. We should specify its domain of definition, but of course this depends very strongly on the boundary conditions, which we have not yet specified. The associated principal eigenfunction ψ satisfies in particular

$$-\nabla \cdot (e^{\varepsilon m} \nabla \psi) - m e^{\varepsilon m} \psi = \lambda_\varepsilon(m) \psi e^{\varepsilon m} \quad \text{in } \Omega,$$

completed by adequate boundary conditions. Following the approach developed in [26], optimal configurations of resources correspond to the ones ensuring the fastest convergence to the steady-states of the PDE above, which comes to minimizing $\lambda_\varepsilon(m)$ with respect to m.

As previously, we will distinguish between two standard boundary conditions on the eigenfunction ψ: Dirichlet and Neumann ones, for which the corresponding eigenvalues are respectively denoted with a D or N superscript. The resulting optimization problems read

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda_\varepsilon^D(m) \quad \text{and} \quad \inf_{m \in \mathcal{M}(\Omega)} \lambda_\varepsilon^N(m) \tag{8}$$

where

$$\lambda_\varepsilon^D(m) = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\frac{1}{2}\int_\Omega e^{\varepsilon m} |\nabla u|^2 - \int_\Omega m e^{\varepsilon m} u^2}{\int_\Omega e^{\varepsilon m} u^2} \quad \text{(Dirichlet case)}$$

$$\lambda_\varepsilon^N(m) = \inf_{u \in W^{1,2}(\Omega)} \frac{\frac{1}{2}\int_\Omega e^{\varepsilon m} |\nabla u|^2 - \int_\Omega m e^{\varepsilon m} u^2}{\int_\Omega e^{\varepsilon m} u^2} \quad \text{(Neumann case).} \tag{9}$$

It is notable that the results strongly differ depending on if one considers the one-dimensional case ($\Omega = (0,1)$) or the multi-dimensional one.

Theorem 9 (1D case, [10])**.** *Let $\Omega = (0,1)$, $\kappa > 0$, $m_0 \in (0, \kappa)$ and $\varepsilon \geq 0$. The optimal design problem (8) has a solution m^*. Moreover, there exists $\varepsilon_0 > 0$ depending on the problem parameters such that for all $\varepsilon < \varepsilon_0$:*

- *in the Dirichlet case, there is a unique minimizer given by $m^* = \kappa \mathbb{1}_{((1-m_0)/2,(1+m_0)/2)}$.*
- *in the Neumann case, there are exactly two minimizers given by $m^* = \kappa \mathbb{1}_{(0,m_0)}$ and $m^* = \kappa \mathbb{1}_{(1-m_0,1)}$.*

In the 1D case, one recovers similar results to the ones obtained in Proposition 1 corresponds to the case $\varepsilon = 0$, whereas we do not expect existence in the higher dimensional cases, except if Ω is a ball.

Theorem 10 (Multi-dimensional case, Dirichlet conditions, [34])**.** *Let Ω be a bounded connected subset of $\mathbb{R}^n$ with a connected Lipschitz boundary, let $\varepsilon > 0$ and $n \geq 2$. Let us consider the Dirichlet case. If the optimization problem (8) has a solution m^*, then it necessarily writes $m^* = \kappa \mathbb{1}_{E^*}$, where E^* is a measurable subset of Ω. Moreover, if ∂E^* is a C^2 hypersurface and if Ω is connected, then Ω is a ball.*

Theorem 10 can be interpreted as follows: assuming that the population density moves along the gradient of the resources, it is not possible to lay the resources in an optimal way. In the 1D case, optimal configurations for more general boundary conditions of Robin type (including the case of homogeneous Dirichlet and Neumann boundary conditions) have been obtained, by using a new rearrangement technique. Finally, let us mention the related result [19, Theorem 2.1], dealing with Faber-Krahn type inequalities for general elliptic operators involving a drift term.

4.2. **Open problems.** We conclude this section by formulating open problems, interesting from the theoretical or numerical point of view, that remain to be investigated or developed.

In what follows, let $\kappa > 0$, $m_0 \in (0, \kappa)$ and E^* be an optimizer for Problem (**I**).

Open problem 1. *In the Neumann case, investigate the validity of the property: "let Ω satisfy ($\mathcal{H}_{\text{reg}}$), then one has $E^* \cap \partial\Omega \neq \emptyset$."*

This conjecture is supported by the analysis of the particular case where Ω is a square (Theorem 3) and numerical computations (see for instance the figures 4, 6 and 8).

Open problem 2. *In the Neumann case, if Ω is a ball, can $\partial E^* \cap \Omega$ be a piece of sphere? Note that if Ω is a square in dimension 2, we already know that $\partial E^* \cap \Omega$ is not a piece of sphere according to Theorem 3, in spite of the numerical results on Figure 4 highlighting that the curvature of several optimizers seems to be almost constant.*

Open problem 3. *Investigate convergence properties of Algorithm 1. In particular:*

- *Can one ensure that, starting from any initial configuration E_0 in Ω, Algorithm 1 converges to a local minimizer for Problem (**I**)?*
- *In that case, can the convergence speed be identified?*

The following open problems are related to Problem (**II**).

Open problem 4. *For general, possibly smooth, open connected domains Ω, obtain a sharp estimate of the nonnegative number μ^* introduced in Theorem 4.*

According to the main result of [37], we conjecture that maximizers are bang-bang functions for any $\mu > 0$, in other words that $\mu^* = 0$.

Open problem 5. *For general, possibly nonsmooth, and non-connected domains Ω, can the main results/approaches of the present article be generalized? In any case, can one numerically look for maximizers with the help of a dedicated shape optimization algorithm?*

Open problem 6. *Let $\Omega = (0,1)$. Given that, for $\mu > 0$ small enough, the optimal configurations for $\lambda_1(\cdot,\mu)$ and F_μ do not coincide according to Proposition 1 and Theorem 6, it would be natural and biologically relevant to investigate the maximization of a convex combination of F_μ and $\lambda_1(\cdot,\mu)$ over $\mathcal{M}(\Omega)$.*

Open problem 7. *For general, possibly smooth, open connected domains Ω, investigate the asymptotic behavior of maximizers for Problem (II) as the parameter μ decreases to 0. Such an issue appears intricate since it requires a refine study of singular limits for the involved operators.*

In the spirit of the problems studied in this chapter, it would be very interesting to study the case of similar optimal control problems for (non-scalar) reaction-diffusion systems. Few results exist in this field. The study of such problems generally poses an obvious difficulty: the usual tools of the "principle of comparison" type generally fail. Moreover, it is not always easy to establish usable persistence criteria.

Let us state hereafter a generic kind of such problems.

Open problem 8. *Let Ω be a bounded connected open set. Formally, let $n_1(t)$ denote a density of wild individuals (typically predators whose evolution needs to be controlled) and $n_2(t)$ the density at time t of a controlled population that we introduced in the environment Ω where the first population lives. For $\mu_1 > 0$ and $\mu_2 > 0$, we consider as model of population density dynamics the competitive compartmental system*

$$\begin{cases} \dfrac{\partial n_1}{\partial t}(t,x) - \mu_1 \Delta n_1(t,x) = f_1(n_1(t,x), n_2(t,x)) & (t,x) \in \mathbb{R}_+ \times \Omega \\ \dfrac{\partial n_2}{\partial t}(t,x) - \mu_2 \Delta n_2(t,x) = f_2(n_1(t,x), n_2(t,x)) + u(t,x) & (t,x) \in \mathbb{R}_+ \times \Omega \\ n_1(0,\cdot) = n_1^0(\cdot), \quad n_2(0,\cdot) = n_2^0(\cdot), & \text{in } \Omega, \end{cases} \tag{10}$$

where u is a non-negative function standing for a control. Hence, a typical issue consists in acting on the second population to steer (n_1, n_2) as close as possible to a stable steady-state (n_1^*, n_2^*) of the system without control. If the horizon of time $T > 0$ is given, we are then led to minimize the distance of (n_1, n_2) to either (n_1^*, n_2^*) or its attraction basin, by considering admissible controls $u \in \mathcal{M}_0(\Omega)$. A standard important issue to derive optimal control strategies is a make the control structure precise. Typically, in what case can one ensure that optimal controls (whenever they exist) are bang-bang?

It is notable that such a problem is a simplified control model of population replacement strategies, where one aims at controlling a population of wild *Aedes* mosquitoes by means of *Wolbachia* infected ones. For this problem, the functions f_i, $i = 1, 2$, are defined by

$$f_1(n_1, n_2) = b_1 n_1 \left(1 - s_h \frac{n_2}{n_1 + n_2}\right)\left(1 - \frac{n_1 + n_2}{K}\right) - d_1 n_1, \tag{11}$$

$$f_2(n_1, n_2) = b_2 n_2 \left(1 - \frac{n_1 + n_2}{K}\right) - d_2 n_2, \tag{12}$$

where s_h, K, b_i and d_i, $i = 1, 2$ denote positive constants. In [1], a close problem, where the criterion is a least square functional and the steady-state corresponds to the extinction of the first population, has been investigated.

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